Appendix: Some mathematical relations for photo analysis

We consider the analysis of the photo of a scenery illuminated by a single source at infinity; that is, all light rays illuminating the scene have the same direction, which we quantify by the unit vector $\mathbf{n}$.

The camera optics are modeled in the geometric optics approximation. The equivalent optical system of the lens is represented by 2 principal planes separated by a certain distance and the image focal length, as shown in Fig. 1, where the coordinate systems used are also shown. All positions in 3-D space are described by the (x, y, z) coordinates with origin at the front principal point. The corresponding 2-D positions on the film are given in terms of the coordinates (X, Y) o a plane that is at a distance f (focal length) from the rear principal plane.

The basic relation resulting from geometric optics determines the position (X, Y) on the film of the image of a sufficiently distant point (so that its image can be considered as located at the focal plane of the lens) whose 3-D position is (x, y, z) as

$$X = -f\frac{x}{z},$$
$$Y = -f\frac{y}{z}.$$

Since we assume that all light rays illuminating the scene have the same direction $\mathbf{n}$, given the point (x_o, y_o, z_o) of a generic object, each corresponding point of its shadow must satisfy

$$(x_s, y_s, z_s) = (x_o, y_o, z_o) + \lambda\mathbf{n},$$

for some value of the parameter λ.

The images of those points on the film are given by

$$(X_o, Y_o) = -f\left(\frac{x_o}{z_o}, \frac{y_o}{z_o}\right).$$
$$(X_s, Y_s) = -f\left(\frac{x_o + \lambda n_x}{z_o + \lambda n_z}, \frac{y_o + \lambda n_y}{z_o + \lambda n_z}\right).$$

In this way, if one identifies the images of a point and its shadow, the line on the film connecting them has the equation, with parameter η,

$$(X, Y) = (X_o, Y_o) + \eta(X_s - X_o, Y_s - Y_o).$$

In particular, since the front principal point is at the origin of coordinates, for an imaginary point located at this cardinal point there would be a shadow in 3D space such that the image of the shadow would be located at the point

$$(X_p, Y_p) = -f\left(\frac{n_x}{n_z}, \frac{n_y}{n_z}\right). \tag{1}$$

Note that in this case, the line connecting the cardinal point and its shadow in 3D space is transformed into a single point in the film plane with coordinates (X_p, Y_p).

The proof that all the lines that connect the points (X_o, Y_o) with their corresponding shadows (X_s, Y_s) must converge at (X_p, Y_p) is simply given by noting that the condition for this to happen,

$$(X_p, Y_p) = (X_o, Y_o) + \eta(X_s - X_o, Y_s - Y_o),$$

for some η, is explicitly written as

$$\frac{X_p - X_o}{X_s - X_o} = \frac{Y_p - Y_o}{Y_s - Y_o},$$

that is,

$$\frac{\frac{n_x}{n_z} - \frac{x_o}{z_o}}{\frac{x_o+\lambda n_x}{z_o+\lambda n_z} - \frac{x_o}{z_o}} = \frac{\frac{n_y}{n_z} - \frac{y_o}{z_o}}{\frac{y_o+\lambda n_y}{z_o+\lambda n_z} - \frac{y_o}{z_o}},$$

which is readily seen to be an identity for any (x_o, y_o, z_o), λ and $\mathbf{n}$.

In this way, even if the shadow of the front principal point does not show in the picture, its position (X_p, Y_p) on the film can be determined as the converging point of the lines connecting points with their shadows.

Besides, once the point (X_p, Y_p) is determined, the 3-D direction of the light can be obtained from relation (1) as

$$(n_x, n_y, n_z) = \pm\frac{(-X_p, -Y_p, f)}{\sqrt{X_p^2 + Y_p^2 + f^2}}, \tag{2}$$

in which the correct sign is simply established by inspection of the photo to determine the sign of n_z.

In the special case $n_z = 0$, the previous relations are still valid, only that the image of the front principal point shadow is at infinity, and all lines connecting a point with its shadow are parallel in the film. The direction of these lines coincides with the direction of the light rays in 3-D space.

As a final observation of practical relevance one must take into account that the previous analysis concerns the image on the actual film, so that if one studies the developed pictures, one must revert to the original image. This process involves the reflection of the axes (X, Y) and a global length scaling.

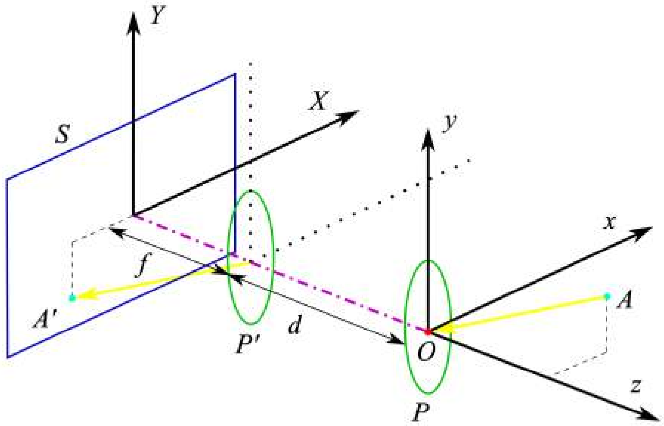

Figure 1: Systems of coordinates used in the analysis. Diagram showing S (blue plane), the film plane; P and P' (green planes), the front and rear principal planes, respectively; O, the front principal point (red circle); the magenta dashed line, the optical axis; f the focal length; d the distance between principal planes; and, A a point in 3D space and A' its corresponding image in the film plane.